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Classification of Bianchi Cosmologies in Conformal Flat Space–Times

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There exist nine types of Bianchi cosmologies classified according to the structure constants of the corresponding Lie groups. Each of these types gives rise to a particular form of the line element, the Friedmann universe corresponding to the simplest type I. It is also known that there exists a simple correspondence (transformation) between the Robertson–Walker line element and the conformal line element but restricting the arbitrary function of that line element. This suggests that a classification of conformal flat line elements according to their parameters should yield a classification similar to that of Bianchi. The conformal group has 15 parameters, corresponding to the pure conformal group, Lorentz group, translation, and dilation. A classification of the line element according to these has been carried out, singly and combining several of them. It has been found that the Friedmann universe is a subclass, as expected, with other cosmologies resulting as wider subclasses. Comparison with the Bianchi classification is also made.

1. INTRODUCTION

The classification of general spaces V_n under a continuous group of transformations G_r was first carried out by Bianchi (1897). The Lie group G_r is characterized by r parameters and defined by the infinitesimal transformations (cf. Eisenhart, 1935)

$$X_\alpha F = \xi_\alpha^\mu \frac{\partial}{\partial x^\mu} \tag{1.1}$$

satisfying the conditions

$$(X_\mu, X_\nu)F = C_{\mu\nu}^\lambda X_\lambda F \tag{1.2}$$

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where $C^{\lambda}_{\mu\nu} = -C^{\lambda}_{\nu\mu}$ are the structure constants obeying the Jacobi identities

$$C^{\mu}_{\alpha\beta}C^{\lambda}_{\mu\nu} + C^{\mu}_{\beta\nu}C^{\lambda}_{\mu\alpha} + C^{\mu}_{\nu\alpha}C^{\lambda}_{\mu\beta} = 0 \qquad (\alpha,\beta,\lambda,\mu,\nu = 1,2,\ldots,r)$$

For the fundamental form

$$\Phi = g_{\mu\nu}dx^{\mu}dx^{\nu}$$

to be the same function of the new coordinates x^{μ} under the infinitesimal transformation it is necessary that Killing's equations (cf. Eisenhart, 1964)

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0 \tag{1.3}$$

be satisfied. It is the solution of these equations that was studied by Bianchi and gave rise to the well-known nine Bianchi types (Taub, 1951). Of particular interest is type I, which yields a constant curvature in the underlying 3-space

$$d\sigma^2 = g_{ij}dX^i dX^j$$

as it forms the basis of the Friedmann universe

$$ds^2 = dT^2 - R^2(T)d\sigma^2 \tag{1.4}$$

Some time ago Infeld and Schild (1945) derived the form of the line element for an isotropic homogeneous expanding universe using conformally flat coordinates. Starting with a line element conformal to a Minkowski space

$$ds^2 = F\varepsilon_{\mu\nu}dx^{\mu}dx^{\nu} = F(dt^2 - dx^2 - dy^2 - dz^2) \tag{1.5}$$

they showed that the limitations of homogeneity and isotropy limited the form the function F obtained from the appropriate solution of Killing's equation. As a matter of fact, the three permissible forms of $F(r,t)$,

$$F = (1 - s^2/4)^{-2} f[t/(1 - s^2/4)] \tag{1.6a}$$

$$F = f(t) \tag{1.6b}$$

$$F = (1 + s^2/4)^{-2} f[t/(1 + s^2/4)] \tag{1.6c}$$

or

$$F = f(s) \tag{1.6c'}$$

where $s^2 = t^2 - r^2$ and f an arbitrary function of its argument turn out to be equivalent to those of a Friedmann universe corresponding to $k = 1, 0, -1$, as can be verified by carrying out appropriate transformations (Infeld and Schild, 1945). Of particular interest to us is (1.6c′) for the open universe of a Lobachewski space (Wess, 1960), where F is only a function of

$$s^2 = \varepsilon_{\mu} x^{\mu} x^{\nu}$$

characteristic of the Poincaré (Lorentz) group.

Thus there exists a one-to-one correspondence between the Bianchi type I and the particular form of the conformally flat line element of Infeld and Schild. This suggests that it might be of interest to find general solutions of Killing's equations corresponding to a conformally flat line element of the form (1.5), but dropping the additional restrictions of isotropy and homogeneity.

In Section 2 of this paper a solution of these equations have been found together with the limitation on the line element, i.e., the function F. It is not surprising that these are the ones corresponding to the conformal group of transformations (Wess, 1960). This group is characterized by 15 parameters and its infinitesimal transformations can be thought to be made up of the following: (a) proper conformal transformations,

$$x^{\mu'} = x^{\mu} + a^{\mu} s^2 - 2x^{\mu}(a_{\alpha} x^{\alpha}) \qquad \text{where } s^2 = \varepsilon_{\mu\nu} x^{\mu} x^{\nu}$$

(b) Lorentz rotations,

$$x^{\mu'} = x^{\mu} + b_{\alpha\beta} \varepsilon^{\beta\mu} x^{\alpha} \qquad \text{with } b_{\alpha\beta} = -b_{\beta\alpha}$$

(c) dilations or scale transformations,

$$x^{\mu'} = x^{\beta} + c x^{\mu}$$

(d) translations,

$$x^{\mu'} = x^{\mu} + e^{\mu}$$

The group is isomorphic to $SO(4,2)$ and the commutation relations between the generators reciprocal to the conformal group

$$C^{\mu} a_{\mu}, \qquad M^{\mu\nu} b_{\mu\nu}, \qquad S\, c, \qquad P^{\mu} e_{\mu}$$

are

$$(C^\mu, C^\nu) = 0$$

$$(M^{\mu\nu}, M^{\lambda\sigma}) = \varepsilon^{\mu\sigma} M^{\nu\lambda} + \varepsilon^{\nu\lambda} M^{\mu\sigma} - \varepsilon^{\mu\lambda} M^{\nu\sigma} - \varepsilon^{\nu\sigma} M^{\mu\lambda}$$

$$(C^\lambda, M^{\mu\nu}) = \varepsilon^{\lambda\mu} C^\nu - \varepsilon^{\lambda\nu} C^\mu$$

$$(C^\lambda, S) = 0, \qquad (C^\lambda, M^{\mu\nu}) = \varepsilon^{\lambda\mu} C^\nu - \varepsilon^{\lambda\nu} C^\mu$$

$$(P^\mu, P^\nu) = 0, \qquad (C^\mu, P^\nu) = 2(M^{\mu\nu} - \varepsilon^{\mu\nu} S)$$

$$(S, P^\nu) = P^\nu, \qquad (S, M^{\mu\nu}) = 0$$

From these the structure constants can easily be deduced and the solutions classified. In Section 3 various combinations are considered and the form of the function F, defining the line element, obtained in these cases. Finally, in Section 4 it is shown how under suitable transformations these solutions can be brought into commonly used forms, of which (1.4) is a special case.

2. KILLING'S EQUATION

For a conformal line element of the form (Tauber, 1967)

$$ds^2 = e^\Gamma \varepsilon_{\mu\nu} dx^\mu dx^\nu \tag{2.1}$$

Killing's equation (1.3) becomes

$$\varepsilon_{\alpha\lambda} \xi^\lambda{}_{,\beta} + \varepsilon_{\beta\lambda} \xi^\lambda{}_{,\alpha} + \varepsilon_{\alpha\beta} \xi^\lambda{}_{,\lambda} = 0 \tag{2.2}$$

which can be written more conveniently as

$$\bar{\xi}_{\alpha,\beta} + \bar{\xi}_{\beta,\alpha} = -\varepsilon_{\alpha\beta} X\Gamma \tag{2.2a}$$

by setting

$$\bar{\xi}_\alpha = \varepsilon_{\alpha\lambda} \xi^\lambda$$

and introducing the Killing operator X defined by

$$X = \xi^\lambda \partial_\lambda$$

Depending on whether $\alpha = \beta$ or $\alpha \neq \beta$ we can now divide (2.2a) into two sets. If $\alpha \neq \beta$ we obtain

$$\bar{\xi}_{\alpha,\beta} + \bar{\xi}_{\beta,\alpha} = 0 \tag{2.3a}$$

while for $\alpha=\beta$ we have

$$\bar{\xi}_{\bar{\alpha},\bar{\alpha}} = -\tfrac{1}{2}\varepsilon_{\alpha\alpha}\overline{\mathbf{X}}\Gamma \qquad \text{(no summation)} \tag{2.3b}$$

from which one deduces that

$$\bar{\xi}_{1,1}=\bar{\xi}_{2,2}=\bar{\xi}_{3,3}=-\bar{\xi}_{4,4}=\phi \tag{2.4}$$

with

$$\overline{\mathbf{X}}\Gamma=2\phi \tag{2.4a}$$

Altogether we have a system of ten equations, nine of which determine the Killing vectors, while the last provides a limitation on the function.

In order to solve (2.2a) consider first the set (2.3a). Differentiating with respect to x^γ $(\gamma\neq\alpha\neq\beta)$ we find

$$\bar{\xi}_{\alpha,\beta\gamma} = -\bar{\xi}_{\beta,\alpha\gamma}$$

Similarly, from a set with β and γ interchanged we obtain

$$\bar{\xi}_{\alpha,\gamma\beta} = -\bar{\xi}_{\gamma,\alpha\beta}$$

and hence

$$\bar{\xi}_{\beta,\alpha\gamma} = \bar{\xi}_{\gamma,\alpha\beta}$$

On the other hand, it follows from (2.2a) that

$$\bar{\xi}_{\beta,\gamma\alpha} = -\bar{\xi}_{\gamma,\beta\alpha}$$

These two sets can only be consistent, provided that

$$\bar{\xi}_{\gamma,\alpha\beta}=0 \tag{2.5}$$

with similar results for other combinations of subscripts. Integrating (2.5) we find

$$\bar{\xi}_{\gamma,\alpha}=f_1(\alpha,\gamma,\delta)$$

where $f_1(\alpha,\gamma,\delta)=f_1(x^\alpha,x^\gamma,x^\delta)$ is an arbitrary function of these variables and $\delta\neq\gamma\neq\beta\neq\alpha$. Interchanging β and δ and using the same considerations leads to

$$\bar{\xi}_{\gamma,\alpha}=f_2(\alpha,\gamma,\beta)$$

Hence

$$\bar{\xi}_{\gamma,\alpha} = f(\alpha,\gamma) \tag{2.5a}$$

Integrating (2.5a) with respect to x^{α} then gives

$$\bar{\xi}_{\gamma} = F(\alpha,\gamma) + G(\beta,\gamma,\delta)$$

where $F(\alpha,\gamma)$ and $G(\beta,\gamma,\sigma)$ are arbitrary functions of their arguments. Differentiating the last equation with respect to x^{β} yields

$$\bar{\xi}_{\gamma,\beta} = G,_{\beta}$$

On the other hand, in analogy with (2.5a) we have

$$\bar{\xi}_{\gamma,\beta} = g(\beta,\gamma) \tag{2.5b}$$

Thus $G(\beta,\gamma,\delta)$ must be of the form

$$G = F_2(\beta,\gamma) + F_3(\gamma,\delta)$$

and $\bar{\xi}_{\gamma}$ itself

$$\bar{\xi}_{\gamma} = F_1(\alpha,\gamma) + F_2(\beta,\gamma) + F_3(\delta,\gamma) \tag{2.6a}$$

Similarly, we obtain

$$\bar{\xi}_{\alpha} = H_1(\beta,\alpha) + H_2(\gamma,\alpha) + H_3(\delta,\alpha) \tag{2.6b}$$

However,

$$\bar{\gamma}_{,\alpha} = -\bar{\xi}_{\alpha,\gamma}$$

which yields

$$F_{1,\alpha} = -H_{2,\gamma}$$

This enables us to write

$$F_1 = \partial K/\partial x^{\gamma} \qquad H_2 = -\partial K/\partial x^{\alpha}$$

This can be done for each pair of indices, resulting finally in the following set of equations

$$\bar{\xi}_1 = \partial_1[A(1,2) + B(1,3) + C(1,4)] \tag{2.7a}$$

$$\bar{\xi}_2 = \partial_2[-A(1,2) + D(2,3) + E(2,4)] \tag{2.7b}$$

$$\bar{\xi}_3 = \partial_3[-B(1,3) - D(2,3) + F(3,4)] \tag{2.7c}$$

$$\bar{\xi}_4 = \partial_4[-C(1,4) - E(2,4) - F(3,4)] \tag{2.7d}$$

where $A(1,2) = A(x^1, x^2)$ etc. as solutions of (II.3a).

We now turn to the set (II.3b). Differentiating, for example,

$$\bar{\xi}_{1,1} = \bar{\xi}_{2,2}$$

with respect to x^1 and using (2.3a) we find

$$\bar{\xi}_{1,11} = \bar{\xi}_{2,21} = \bar{\xi}_{2,12} = \bar{\xi}_{1,22}$$

Substituting (2.7a) we find

$$\partial_1\{\partial_1\partial_1[A(1,2) + B(1,3) + C(1,4)] + \partial_2\partial_2 A(1,2)\} = 0$$

from which we conclude, for example, that $(\partial_1)^3 B(1,3)$ is independent of x^3. In a similar fashion, using

$$\bar{\xi}_{2,23} = \bar{\xi}_{2,33} = -\bar{\xi}_{3,22}$$

we find that $(\partial_3)^3 B(1,3)$ is independent of x^1. Hence we can write

$$\begin{aligned} B(1,3) = {} & B_0(x^3) + B_{11}x^1x^3 + B_{12}x^1(x^3)^2 \\ & + B_{21}(x^1)^2x^3 + B_{22}(x^1)^2(x^3)^2 + B_3(x^1) \end{aligned} \tag{2.8}$$

where B_{ij} $(i,j = 1,2)$ are constants and B_0, B_3 arbitrary functions of their arguments. In the same way analogous relations for the other functions $A(1,2)$, $C(1,4)$, etc. are obtained. Substituting these into (2.7) yields

$$\begin{aligned} \bar{\xi}_1 = {} & f_1(x^1) + A_{11}x^2 + B_{11}x^3 + C_{11}x^4 + A_{12}(x^2)^2 + B_{12}(x^3)^2 + C_{12}(x^4)^2 \\ & + 2x^1\left[A_{21}x^2 + B_{21}x^3 + C_{21}x^4 + A_{22}(x^2)^2 + B_{22}(X^3)^2 + C_{22}(X^4)^2\right] \\ \bar{\xi}_2 = {} & f_2(x^2) - A_{11}x^1 + D_{11}x^3 + E_{11}x^4 - A_{21}(x^1)^2 + D_{12}(x^3)^2 + E_{12}(x^4)^2 \\ & + 2x^2\left[-A_{12}x^1 + D_{21}x^3 + E_{21}x^4 - A_{22}(x^4)^2 + D_{22}(x^2)^2 + F_{22}(x^4)^2\right] \\ \bar{\xi}_3 = {} & f_3(x^3) - B_{11}x^1 - D_{11}x^2 + F_{11}x^4 - B_{21}(x^1)^2 - D_{21}(x^2)^2 + F_{12}(x^4)^2 \\ & + 2x^3\left[-B_{12}x^1 - D_{12}x^2 + F_{21}x^4 - B_{22}(x^2)^2 + F_{22}(x^4)^2\right] \end{aligned} \tag{2.9}$$

$$\bar{\xi}_4 = f_4(x^4) - C_{11}x^1 - E_{11}x^2 - F_{11}x^3 - C_{21}(x^1)^2 - E_{21}(x^2)^2 - F_{21}(x^3)^2$$
$$+ 2x^4\left[- C_{12}x^1 - E_{12}x^2 - F_{12}x^3 - C_{22}(x^1)^2 - E_{22}(x^2)^2 - F_{22}(x^3)^2 \right]$$

If we now calculate $\xi_{\bar{\alpha},\bar{\alpha}}$ and use (2.3b) directly we find that

$$A_{22} = B_{22} = C_{22} = D_{22} = E_{22} = F_{22} = 0$$

as well as

$$B_{12} = -C_{12} = A_{12} = a_1$$
$$D_{12} = -E_{12} = -A_{21} = a_2$$
$$D_{21} = F_{12} = B_{21} = -a_3$$
$$E_{21} = F_{21} = C_{21} = -a_4$$

while the functions f_α must be of the form

$$f_1 = -a_1(x^1)^2 - cx^1 - e^1$$
$$f_2 = -a_2(x^2)^2 - cx^2 - e^2$$
$$f_3 = -a_3(x^3)^2 - cx^3 - e^3$$
$$f_4 = -a_4(x^4)^2 - cx^4 - e^4$$

where a_μ, e^μ ($\mu = 1$–4), and c are constants.

If we now denote the remaining coefficients $A_{11}, B_{11}, \ldots, F_{11}$ by

$$A_{11} = b_{21}, \qquad B_{11} = b_{31}, \qquad C_{11} = b_{41},$$
$$D_{11} = b_{32}, \qquad E_{11} = b_{42}, \qquad F_{11} = b_{43}$$

and introduce the invariant

$$s^2 = \varepsilon_{\alpha\beta}x^\alpha x^\beta = (x^4)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

the Killing vector can be written as

$$\bar{\xi}_\alpha = 2a_\mu x^\mu \varepsilon_{\alpha\beta}x^\beta - s^2 a_\alpha - b_{\alpha\mu}x^\mu + c\varepsilon_{\alpha\beta}x^\beta + \varepsilon_{\alpha\beta}e^\beta \qquad (2.10)$$

or

$$\xi^\alpha = 2a_\mu x^\mu x^\alpha - s^\alpha \varepsilon^{\beta\alpha} a_\alpha + b_{\mu\beta}\varepsilon^{\beta\alpha}x^\mu + cx^\alpha + e^\alpha \qquad (2.10a)$$

which is exactly the form one would have expected from the conformal group characterized by the 15 parameters a_μ, $b_{\mu\nu}$ $(= -b_{\nu\mu})$, e^α, and c. The Killing operator X then becomes

$$\bar{\mathbf{X}} = 2a_\mu x^\mu x^\alpha \partial_\alpha - s^2 \varepsilon^{\beta\alpha} a_\beta \partial_\alpha + b_{\mu\beta} \varepsilon^{\beta\alpha} x^\mu \partial_\alpha + c x^\alpha \partial_\alpha + e^\alpha \partial_\alpha \tag{2.11}$$

Finally, inserting (2.10a) and (2.11) into (2.4a) gives

$$\mathbf{X}\Gamma = -4a_\mu x^\mu - 2c \tag{2.12}$$

as the limitation on the conformal line element. It is the solution of this equation which will give rise to the various types and which will be considered in the following section.

3. OPERATOR EQUATION

We now turn to the operator equation (2.12) whose solution determines the possible form of the line element (2.1). To classify the solutions it is convenient to separate the operator X into its various constituents by writing

$$X = \sum_{i=1}^{4} L_i \tag{3.1}$$

where

$$L_1 = 2\alpha_\mu x^\mu x^\lambda \partial_\lambda - s^2 \varepsilon^{\lambda\alpha} a_\lambda \partial_\alpha \tag{3.1a}$$

will determine the solution for the proper contact transformation,

$$L_2 = b_{\mu\beta} \varepsilon^{\beta\alpha} x^\mu \partial_\alpha \tag{3.1b}$$

those for the Lorentz transformation,

$$L_3 = c x^\alpha \partial_\alpha \tag{3.1c}$$

describes the dilation, and finally

$$L_4 = e^\lambda \partial_\lambda \tag{3.1d}$$

gives those corresponding to translation. The types of solutions which we shall obtain will correspond to one or more of these operators, and thus provide us with a system of classification.

The function Γ can be expanded in terms of polynomials of the form

$$\Gamma = \sum_i A^i_\mu x^\mu + \sum_i A^i_{\alpha\beta} x^\alpha x^\beta + \sum_i A^i_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma + \cdots \tag{3.2}$$

or more compactly as

$$\Gamma = \Gamma(\zeta^i, \eta^i, \xi^i, \ldots) \tag{3.2a}$$

with

$$\zeta^i = A^i_\mu x^\mu, \qquad \eta^i = A^i_\alpha x^\alpha x^\beta, \qquad \xi^i = A^i_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma$$

For simplicity's sake, we shall limit us to the first two terms only, for which

$$\Gamma = \Gamma(\zeta, \eta) \tag{3.3}$$

where

$$\zeta = A_\mu x^\mu, \qquad \eta = A_{\alpha\beta} x^\alpha x^\beta \tag{3.3a}$$

In addition to these two functions we shall also require

$$\phi = \alpha_\mu x^\mu, \qquad \theta = e_\lambda x^\lambda, \qquad s^2 = \varepsilon_{\mu\nu} x^\mu x^\nu \tag{3.3b}$$

and list in Table I various relations and constants arising from them.

We are now in the position to consider solutions of (2.12) for a function of the form of (3.3). Although, of course, it is the full conformal group which is of particular interest, for the sake of completeness and basis for a classification we shall consider all different combinations of the operators occurring in (3.1). To save space only the results are given, listed in the Appendix.

TABLE I. Functions and relations.[a]

Function	ζ	η	ϕ	θ	s
Definition	$\zeta = A_\mu x^\mu$	$\eta = A_{\mu\nu} x^\mu x^\nu$	$\phi = \alpha_\mu x^\mu$	$\theta = e_\mu x^\mu$	$s^2 = \varepsilon_{\mu\nu} x^\mu x^\nu$
Derivative	$\zeta_\lambda = A_\lambda$	$\eta_\lambda = 2A_{\lambda\mu} x^\mu$	$\phi_\lambda = a_\lambda$	$\theta_\lambda = e_\lambda$	$s_\lambda = s^{-1} \varepsilon_{\lambda\mu} x^\mu$
Contraction with x^λ	$x^\lambda \zeta_\lambda = \zeta$	$x^\lambda \eta_\lambda = 2\eta$	$x^\lambda \phi_\lambda = \phi$	$x^\lambda \theta_\lambda = \theta$	$x^\lambda s_\lambda = s$
Contraction with a^λ	$\alpha^\lambda \zeta_\lambda = M$	$\alpha^\lambda \eta_\lambda = 2B_\mu x^\mu$	$a^\lambda \phi_\lambda = a^2$	$a^\lambda \theta_\lambda = \varepsilon$	$a^\lambda s_\lambda = \phi/s$
Contraction with e_λ	$e^\lambda \zeta_\lambda = N$	$e^\lambda \eta_\lambda = 2C_\mu x^\mu$	$e^\lambda \phi_\lambda = \varepsilon$	$e^\lambda \theta_\lambda = e^2$	$e^\lambda s_\lambda = \theta/s$

[a]Where $B_\mu = a^\lambda A_{\lambda\mu}$, $C_\mu = e^\lambda A_{\lambda\mu}$, M, N, a^2, ε, and e^2 are constants.

Even a brief examination of these solutions indicates their general behavior. We note first that the conformal transformation only specifies the argument but not the form of the solution. On the other hand, under a dilation the form is also given. Thus a combination of the two will specify both the argument and nature of the solution. Any function of s is invariant under a Lorentz transformation as are other bilinear forms, but there is some difficulty with linear forms, such as $a_\mu x^\mu$ which enter in the proper conformal transformation. In order to overcome this problem it is necessary to assume some relation between the coefficients $b_{\mu\nu}$ and A_μ involving the solution of a secular equation. In the case of pure translations the situation is similar to that of dilation, but also there some assumptions about the coefficients e_μ must be made. In particular if translations are to be combined with, say, proper conformal transformations, this requires that either the vector e_μ is proportional to a_μ, or a null vector, i.e., $e^2=0$, or at least orthogonal to the other vectors entering in the solution. Combining several of these transformations does not necessarily limit the allowed form or argument of the solution, as long as the appropriate conditions are satisfied. In some cases the extension to several (linear) invariants is necessary to specify the solution. For example, for the complete conformal group the solution is expressed in terms of two linear invariants, a function of s and several constants partly determined by solutions of appropriate secular equations. In all cases it has been possible to find a solution, generally a function of s and one or more linear invariants with constants either arbitrary or determined by additional conditions. The relationship of these solutions to the usual form of the line element will be discussed in the next section.

4. COMPARISON WITH EXPANDING UNIVERSES

The Friedman universe (1.4) can be written in the isotropic form

$$ds^2 = dT^2 - \frac{G^2(T)}{1+\frac{1}{4}kY^2}\, dY\cdot dY \tag{4.1}$$

where

$$Y^i = X^i / R_0$$

are the dimensionless coordinates and

$$R = R_0 G$$

the radius. If $k=-1$ (open universe) the transformation

$$T = T(s), \qquad Y^i = 2x^i/(t+s) \tag{4.2}$$

reduces (4.1) to (2.1) provided we choose

$$dT/ds = G(T)/s \qquad \text{or} \qquad \ln s = \int dT/G \tag{4.2a}$$

with

$$e^{\Gamma} = G^2/s^2 \tag{4.2b}$$

Thus all solutions of the operator equation (2.12) which are functions of s only correspond to an open Friedman universe. In particular, if

$$\Gamma = -2\ln s$$

as is the case for the special solution [(A.6i) corresponding to a Lorentz transformation with dilation] G will be a constant, i.e., the universe is static. However, different powers of s entering in e^{Γ} will give expanding universes.

Another useful form of the line element (1.4) is given by

$$ds^2 = dT^2 - R^2\left[d\rho^2 + S^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{4.3}$$

where

$$\begin{aligned} S(\rho) &= \sin\rho \qquad (k=1) \\ &= \sinh\rho \qquad (k=-1) \\ &= \rho \qquad (k=0) \end{aligned}$$

obtained from (4.1) by the transformation

$$Y = S(\rho)$$

Finally, a transformation of the time coordinate

$$\tau = \int dT/R(T), \qquad R(T) = R(\tau)$$

results in a completely isotropic form

$$ds^2 = R^2\left[d\tau^2 - d\rho^2 - S^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{4.3a}$$

At the same time the conformal line element (2.1) may be expressed in terms of spherical polar coordinates as

$$ds^2 = e^{\Gamma}\left[dt^2 - dr^2 - r^2(d\alpha^2 + \sin^2\alpha d\beta^2)\right] \tag{4.4}$$

which permits an easy comparison with (4.3).

If Γ is a function of ζ only, such as in the case of the proper conformal transformation with translation [Appendix, part (9)] the time variable in (4.4) may be replaced by ζ, resulting in

$$ds^2 = e^{\Gamma(\zeta)}\left[A_4^{-2}d\zeta^2 - dr^2 - r^2(d\alpha^2 + \sin^2\alpha d\beta^2)\right] \tag{4.5}$$

The further transformation

$$dT/d\zeta = e^{(1/2)\Gamma}/A_4 \qquad \text{or} \qquad T = \frac{1}{A_4}\int e^{1/2\Gamma}d\zeta$$

brings (4.5) then in a form

$$ds^2 = dT^2 - e^{\Gamma}\left[dr^2 + r^2(d\alpha^2 + \sin^2\alpha d\beta^2)\right] \tag{4.5a}$$

which is identical to (4.3) provided we identify

$$\rho = r, \qquad \theta = \alpha, \qquad \phi = \beta$$

Thus we see that this case simply corresponds to a flat universe. In particular, if

$$\Gamma = -2\ln\zeta$$

as given by (A.9c) the radius $R(T)$ is given by

$$R(T) = R_0 e^{-TA_4/R_0}$$

If the function Γ depends on ζ and $\bar{\zeta}$, or on ζ and s, or even is a function of all three variables, it is advisible to introduce a new system of coordinates

$$\zeta^{\alpha} = A^{\alpha}_{\mu}x^{\mu} \qquad (\alpha = 1,2,3,4) \tag{4.6}$$

with $\zeta^1 = \zeta$ and $\zeta^2 = \bar{\zeta}$ or $\zeta^4 = \zeta$ and $\zeta^1 = \bar{\zeta}$, while the remaining coefficients A^i_{μ} are still unspecified. If the determinant of the matrix A^{α}_{μ} does not vanish, it is possible to find the inverse tranformation

$$x^{\mu} = B^{\mu}_{\alpha}\zeta^{\alpha} \tag{4.6a}$$

where the coefficients B^μ_α are the cofactors of A^μ_α divided by $\det|A^\mu_\alpha|$. The conformal flat line element (2.1) then becomes

$$ds^2 = e^\Gamma \varepsilon_{\mu\nu} dx^\mu dx^\nu = e^\Gamma \varepsilon_{\mu\nu} B^\mu_\alpha B^\nu_\beta d\zeta^\alpha d\zeta^\beta = e^\Gamma E_{\alpha\beta} d\zeta^\alpha d\zeta^\beta \tag{4.7}$$

and, since some of the coefficients A^μ_α are still arbitrary, can be brought into a diagonal form by demanding that

$$E_{\alpha\beta} = \varepsilon_{\alpha\beta} E_\alpha \qquad \text{(no summation)}$$

If, at the same time, we introduce coordinates Z^α defined by

$$Z^\alpha = \zeta^\alpha E_\alpha \qquad \text{(no summation)}$$

we finally obtain

$$ds^2 = e^\Gamma \varepsilon_{\alpha\beta} dZ^\alpha dZ^\beta \tag{4.8}$$

where Γ is now assumed to be a function of the Z^α.

As an example consider the case of a proper conformal transformation with dilation and translation for which Γ is given by [cf. (A.13b)]

$$\Gamma = -2\ln\left(\overline{M}\zeta - M\zeta\right)$$

Setting

$$U = \overline{M}\zeta - M\zeta = \overline{M}\zeta^4 - M\zeta^1$$
$$V = \overline{M}\zeta + M\zeta = \overline{M}\zeta^4 + M\zeta^1 \tag{4.9}$$

the line element (4.7) becomes

$$ds^2 = U^{-2}\left[K(dU^2 + dV^2) + L\,dUdV - E_2(d\zeta^2)^2 - E_3(d\zeta^2)^2\right] \tag{4.10}$$

where

$$K = \frac{1}{4}\left(\frac{E_4}{4\overline{M}^2} - \frac{E_1}{4M^2}\right), \qquad L = \left(\frac{E_4}{\overline{M}} + \frac{E_1}{M}\right)$$

Upon imposing still another condition on the coefficients A^μ_α it is possible to demand that $L = 0$ and thus diagonalizing (4.10). The time T can now be introduced through

$$T = K^{1/2}\ln U \qquad \text{or} \qquad U = e^{T/(K^{1/2})}$$

So that (4.10) has the right signature K must be negative, which is achieved by setting

$$K = -\omega^{-2}$$

If we now also define space coordinates X, Y, and Z through

$$X = V/\omega, \qquad Y = (E_2)^{1/2}\zeta^2, \qquad Z = (E_3)^{1/2}\zeta^3$$

the line element finally takes the form

$$ds^2 = dT^2 - e^{2i\omega T}(dX^2 + dY^2 + dZ^2) \tag{4.10a}$$

We note that we have again a flat universe, but with a radius which varies periodically in time.

On the other hand, had we taken ζ as a space vector as well and set

$$\bar{U} = \bar{M}_\zeta - M_{\bar{\zeta}} = \bar{M}\zeta^1 - M\zeta^2$$
$$\bar{V} = \bar{M}_\zeta + M_{\bar{\zeta}} = \bar{M}\zeta^1 + M\zeta^2 \tag{4.11}$$

the line element (4.7) would have become

$$ds^2 = \bar{U}^{-2}\left[E_4(d\zeta^4)^2 - \bar{K}(d\bar{U}^2 + d\bar{V}^2) - \bar{L}\,dUdV - E_3(d\zeta^3)^2\right. \tag{4.12}$$

with

$$\bar{K} = \frac{1}{4}\left(\frac{E_1}{\bar{M}^2} + \frac{E_2}{M^2}\right), \qquad \bar{L} = \left(\frac{E_1}{\bar{M}} - \frac{E_2}{M}\right)$$

Demanding that $\bar{L} = 0$ and at the same time introducing the coordinates X, Y, Z, and T through

$$X = \bar{K}^{1/2}U, \qquad Y = \bar{K}^{1/2}V, \qquad Z = (E_3)^{1/2}\zeta^3, \qquad T = (E_4)^{1/2}\zeta^4$$

transforms (4.12) into

$$ds^2 = \frac{K}{X^2}(dT^2 - dX^2 - dY^2 - dZ^2) \tag{4.12a}$$

In this form the solution (A.13b) does not correspond to an expanding universe at all, but is a particular static solution of the field equations.

If Γ is a function of ζ and s, as is the case for the proper conformal transformation (and most of the other cases considered) it is convenient to write (4.8) in the spherical symmetric form

$$ds^2 = e^{\Gamma}\left[(dZ^4)^2 - dZ^2 - Z^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{4.13}$$

where

$$Z^1 = Z\sin\theta\cos\phi$$

$$Z^2 = Z\sin\theta\sin\phi$$

$$Z^3 = Z\cos\theta$$

At the same time s^2 now becomes

$$s^2 = \varepsilon_{\mu\nu}x^\mu x^\nu = E_{\alpha\beta}\zeta^\alpha\zeta^\beta = \varepsilon_{\alpha\beta}Z^\alpha Z^\beta = (Z^4)^2 - Z^2 \tag{4.14}$$

This immediately suggests the transformation

$$Z^4 = P + Q, \qquad Z = P - Q \tag{4.15}$$

resulting in

$$ds^2 = e^{\Gamma}\left[4\,dPdQ - (P-Q)^2(d\theta^2 + \sin\theta d\phi^2)\right] \tag{4.15a}$$

To get (4.15) into the form of a closed universe, such as (4.3a), we follow Infeld and Schild (1945) and set

$$P = \tan\pi, \qquad Q = \tan\sigma \tag{4.16}$$

which after some simplification yields

$$ds^2 = e^{\Gamma}\sec^2\pi\sec^2\sigma\left[4d\pi d\sigma - \sin^2(\pi-\sigma)(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{4.16a}$$

Finally, setting

$$\tau = \pi + \sigma, \qquad \rho = \pi - \sigma \tag{4.17}$$

results in

$$ds^2 = 4e^{\Gamma}(\cos\tau + \cos\rho)^{-2}\left[d\tau^2 - d\rho^2 - \sin^2\rho(d\theta^2 + \sin\theta d\phi^2)\right] \tag{4.17a}$$

which corresponds to a closed universe of the form (4.13a), except that the radius R is now a function of ρ as well as τ. Had we used instead of (4.16) the transformation

$$P = \tanh\pi, \qquad Q = \tanh\sigma$$

the final form of the line element would have been

$$ds^2 = 4e^{\Gamma}(\cosh\tau + \cosh\rho)^{-2}\left[d\tau^2 - d\rho^2 - \sinh^2\rho(d\theta^2 + \sin\theta d\phi^2)\right] \tag{4.17b}$$

an open universe, with a radius depending on ρ and τ.

As an example, consider a function Γ which is a function of the argument

$$(\zeta^2/s^2 + C)s^{-2} \tag{4.18}$$

where C is a constant, and of which (A.1a) is a special case. Carrying out the various transformations (4.15), (4.16), and (4.17) finally gives

$$\frac{\sin^2\tau + C^2E_4^2(\cos^2\rho - \cos^2\tau)}{4E_4^2(\cos\rho - \cos\tau)^2} \tag{4.18a}$$

The exact form of the radius R and its dependence on ρ and τ is determined on how Γ depends on its argument (4.18) and will be found only from the field equations. It should be noted that the above analysis works equally well, if Γ is a function of ζ and η, except that now the coefficients $A_{\mu\nu}$ entering in η have also to be taken into account.

In the general case of the complete conformal group, where Γ is a function of ζ and $\bar{\zeta}$, as well as s [cf. (A.15)] we shall at first proceed as in the more special case of proper conformal transformation with dilation and translation [cf. (A.13b)]. To avoid later complications with the signature we shall immediately set

$$\begin{aligned} U &= F\zeta + G\bar{\zeta} = F\zeta^4 + G\zeta^1 \\ iV &= F\zeta - G\bar{\zeta} = F\zeta^4 - G\zeta^1 \end{aligned} \tag{4.19}$$

which transforms the line element (4.17) into

$$ds^2 = e^{\Gamma}\left[K(dU^2 - dV^2) + iLdUdV - E_2(d\zeta^2)^2 - E_3(d\zeta^3)^2\right] \tag{4.19a}$$

where again

$$K = \frac{1}{4}\left(\frac{E_4}{F^2} - \frac{E_1}{G^2}\right), \qquad L = \left(\frac{E_4}{F} + \frac{E_1}{G}\right)$$

This can again be diagonalized by subjugating the coefficients A_α^μ to additional conditions and demanding that $L=0$. If we now introduce new coordinates W, Z, θ, and ϕ defined by

$$W=K^{1/2}U$$
$$K^{1/2}V=Z\sin\theta\cos\phi$$
$$E_2^{1/2}\zeta^2=Z\sin\theta\sin\phi$$
$$E_3^{1/2}\zeta^3=Z\cos\theta$$

the line element (4.19a) becomes

$$ds^2=e^\Gamma\left[dW^2-dZ^2-Z^2(d\theta^2+\sin\theta d\phi^2)\right] \tag{4.19b}$$

which is of the form (4.13).

Proceeding now as before, with W taking the place of Z^4, we end up with a closed (or open) universe of the form (4.17b). However, this time the form of the function Γ and thus the radius R is specified. Writing (A.15) in the form

$$e^\Gamma=\left(F\zeta+G\bar{\zeta}+\tfrac{1}{2}As^2+2C\right)^{-2} \tag{4.20}$$

where the constants A and C are defined by

$$\tfrac{1}{2}A=G\overline{M}+FM, \qquad 2C=G\overline{N}+FN$$

and carrying out the transformations indicated by (4.15), (4.16), (4.17), and (4.19) transforms (4.20) into

$$e^\Gamma=4\left[\sin\tau+\cos\rho(A+C)+\cos\tau(C-A)\right]^2(\cos\rho+\cos\tau)^{-2} \tag{4.20a}$$

Finally, inserting this expression into (4.17b) results in a closed universe of the form (4.3a) with the radius R given by

$$R(\tau,\rho)=\left[\sin\tau+\cos\rho(A+C)+\cos\tau(C-A)\right]^{-1} \tag{4.21}$$

It should be noted that in the special case for which $A+C=0$ we obtain again a closed universe for which the radius R is only a function of τ as in (4.3a).

To complete our analysis let us consider ζ again as a space vector and in analogy with (4.11) set

$$\bar{U} = F\zeta + G\bar{\zeta} = F\zeta^1 + G\zeta^2$$
$$\bar{V} = F\zeta - G\bar{\zeta} = F\zeta^1 - G\zeta^2 \tag{4.22}$$

This transforms the line element (4.7) again into a form closely related to (4.12a):

$$ds^2 = e^{\Gamma}\left[dT^2 - dX^2 - dY^2 - dZ^2\right] \tag{4.22a}$$

with the coordinates defined by

$$T = E_4^{1/2}\zeta^4, \qquad X = K^{1/2}\bar{U}, \qquad Y = K^{1/2}\bar{V}, \qquad Z = E_3^{1/2}\zeta^3$$

At first thought it might be advisable to introduce again spherically symmetric coordinates, but since Γ depends explicitly on X as well as on s this would make it a function of three variables. Instead, we find it advantageous to define the variables

$$T = W\cosh\alpha$$
$$Y = W\sinh\alpha\cos\beta$$
$$Z = W\sinh\alpha\sin\beta \tag{4.23}$$

in terms of which (4.22a) becomes

$$ds^2 = e^{\Gamma}\left[dW^2 - dX^2 - W^2(d\alpha^2 + \sinh^2\alpha d\beta)\right] \tag{4.23a}$$

while s^2 is given by

$$s^2 = T^2 - X^2 - Y^2 - Z^2 = W^2 - X^2 \tag{4.23b}$$

If we now introduce again the variables P and Q defined by

$$X = P + Q, \qquad W = P - Q \tag{4.24}$$

and follow this by transformations (4.16) and (4.17) the line element (4.23a) takes the form

$$ds^2 = 4e^{\Gamma}(\cos\rho + \cos\tau)^{-2}\left[d\tau^2 - d\rho^2 - \sin^2\rho\left(d\alpha^2 + \sinh^2\alpha d\beta^2\right)\right] \tag{4.24a}$$

This is similar to a closed universe, except that the 2-space characterized by the angles α and β is not that of a sphere, but of a hyperboloid of revolution. Expressing again the function e^{Γ} in terms of these variables, we finally obtain

$$ds^2 = \left[\sin\tau + \cos\rho(A+C) + \cos\tau(A-C)\right]^{-2}$$

$$\times\left[d\tau^2 - d\rho^2 - \sin^2\rho\left(d\alpha^2 + \sinh^2\alpha\, d\beta^2\right)\right] \tag{4.24b}$$

From the above examples we may thus conclude that it is always possible to transform a line element corresponding to one or more elements of the conformal group into one resembling a Friedmann universe. In special cases, where the function multiplying the flat line element only depends on ζ (or its special cases) the result will be a flat space; for any function of s it will be an open universe, while in the general case it will be a closed (or open) universe, but with a radius which also depends on one space coordinate. In this sense, the resulting group contains more general line elements than the three Friedmann universes with which we started.

APPENDIX: SOLUTIONS OF THE OPERATOR EQUATION (2.12)

1. *Proper Conformal Transformation*
Equation: $L_1\Gamma = -4\phi$
Special solution:

$$\Gamma_o = -4\ln s \tag{A.1}$$

General solution:

$$\Gamma = \Gamma_o + \bar{\Gamma}(\chi) \tag{A.1a}$$

where

$$\chi = \phi^2 s^{-4} - a^2 s^{-2} \tag{A.1a}$$

2. *Lorentz Transformation*
Equation: $L_2\Gamma = 0$
General solution:

$$\text{a)} \qquad \Gamma = \Gamma(s) \tag{A.2a}$$

$$\text{b)} \qquad \Gamma = \Gamma(\eta) \tag{A.2b}$$

provided that

$$A_{\alpha\beta}=b_{\alpha\mu}\varepsilon^{\mu\nu}b_{\nu\beta} \tag{A.2b'}$$

c) independent of ζ if

$$b_{\alpha\mu}A^{\mu}=0 \tag{A.3}$$

3. *Proper Conformal and Lorentz Transformations*
Equation: $(L_1+L_2)\Gamma=-4\phi$
General solution:

$$\text{a)}\qquad \Gamma=-4\ln s+\bar{\Gamma}(\chi) \tag{A.3a}$$

i) if $b_{\mu\nu}\alpha^{\nu}=0$

$$\chi=\phi^2 s^{-4}+a^2 s^{-2} \qquad \text{(see A.1a')} \tag{A.3ai}$$

ii) if $b_{\mu\nu}\alpha^{\nu}=D_{\alpha\mu}$, which implies $|b_{\mu\nu}-\varepsilon_{\mu\nu}D|=0$ and $a^2=0$, then

$$\chi=\phi s^{-2}+Ds^{-2} \tag{A.3aii}$$

$$\text{b)}\qquad \Gamma=-4\ln s+\bar{\Gamma}(\psi) \tag{A.3b}$$

where $\psi=(\zeta+F)s^{-2}$, provided that

$$b_{\mu\alpha}A^{\alpha}=2Fa_{\mu} \tag{A.3b'}$$

which implies $M=0$.

4. *Dilation*
Equation: $L_3\Gamma=-2c$
General solution:

$$\Gamma=p\ln\zeta+q\ln\eta \tag{A.4}$$

where

$$p+2q=-2 \tag{A.4a}$$

5. *Proper Conformal Transformation with Dilation*
Equation: $(L_1+L_3)\Gamma=-4\phi-2c$
General solution:

$$\Gamma=-4\ln s+\bar{\Gamma} \tag{A.5}$$

$$\text{(a)}\qquad \bar{\Gamma}=-2\ln\chi \tag{A.5a}$$

where $\chi=\phi s^{-2}+a^2c$.

$$\text{(b)} \qquad \bar{\Gamma}=-\ln\bar{\chi} \tag{A.5b}$$

where $\bar{\chi}=(\phi+c/4)s^{-2}+a^2/2c$.

$$\text{(c)} \qquad \bar{\Gamma}=-\tfrac{1}{2}\ln\bar{\bar{\chi}} \tag{A.5c}$$

where $\bar{\bar{\chi}}=(\phi^2+p\phi+q)s^{-4}+(m\phi+n)s^{-2}+t$, with $p=c$, $q=c^2/4$, $m=a^2/c$, $n=\frac{1}{2}a^2$, $t=a^2/4c^2$.

6. *Lorentz Transformation with Dilation*

Equation: $(L_2+L_3)\Gamma=-2c$

Special solution:

$$\text{(i)} \qquad \Gamma=-2\ln s \tag{A.6i}$$

$$\text{(ii)} \qquad \Gamma=-\ln\eta \tag{A.6ii}$$

provided that $A_{\mu\nu}=b_{\mu\alpha}\varepsilon^{\alpha\beta}b_{\beta\nu}$.

General solution:

$$\Gamma=(c+E)\ln\zeta+2c\ln\eta-2\ln s \tag{A.6a}$$

provided that

$$b_{\mu\alpha}\varepsilon^{\alpha\beta}A_\beta=EA_\mu \tag{A.6a$'$}$$

which implies $|b_{\mu\nu}-\varepsilon_{\mu\nu}E|=0$ and $A_\mu A^\mu=0$.

7. *Proper Conformal and Lorentz Transformation with Dilation*

Equation: $(L_1+L_2+L_3)\Gamma=-4\phi-2c$

Solution:

$$\text{(a)} \qquad \Gamma=-4\ln s+\bar{\Gamma} \tag{A.7a}$$

(i) if $b_{\mu\nu}a^\nu=0$, L_2 does not enter, solutions given by (A.5a, b, c); (ii) if $b_{\mu\nu}a^\nu=Da_\mu$, which implies $|b_{\mu\nu}-\varepsilon_{\mu\nu}D|=0$ and $a^2=0$,

$$\bar{\Gamma}=\frac{2c}{D-c}\ln(\phi s^{-2}) \tag{A.7ai}$$

$$\bar{\Gamma}=-\ln\left(\phi s^{-2}+\frac{c+D}{4s^2}\right) \tag{A.7aii}$$

$$\bar{\Gamma}=-\tfrac{1}{2}\ln(\phi^2+\bar{p}\phi+\bar{q})s^{-4} \tag{A.7aiii}$$

where $\bar{p}=c-D$, $\bar{q}=\frac{1}{4}(c-D)^2$.

$$\text{(b)} \qquad \Gamma = A\ln(f\zeta+g) \tag{A.7b}$$

where $A=2c/(E-c)$, $f=s^{-p}$, $g=Gs^{-q}$, $p=(c+E)/c$, $G=\frac{1}{2}M/(2c-E)$, $q=2(E-c)/c$ and $b_{\mu\nu}A^{\nu}=EA_{\mu}$, which implies $|b_{\mu\nu}-\varepsilon_{\mu\nu}E|=0$ and $A_{\mu}A^{\mu}=0$.

8. *Translation*
Equation: $L_4\Gamma=0$
Solution:

$$\Gamma=\frac{k}{2}(\zeta^2/N-\eta/c) \tag{A.8}$$

provided that $e^{\lambda}A_{\lambda\mu}=C_{\mu}=CA_{\mu}$ and k is a constant of integration.

9. *Proper Conformal Transformation with Translation*
Equation: $(L_1+L_4)\Gamma=-4\phi$
Solution:

(a) if $e_{\lambda}=Ea_{\lambda}$:
special solution,

$$\Gamma=f=-2\ln(s^2+E) \tag{A.9a}$$

general solution,

$$\Gamma=f+\Gamma(\chi) \tag{A.9a$'$}$$

where $\chi=A/(s^2+E)^2=(\phi^2-a^2)+B$.

(b) if $e^2=0$:
general solution,

$$\Gamma=-\ln\left(\theta^2+\frac{\epsilon}{2}s^4\right) \tag{A.9b}$$

(c) if

$$\Gamma=-2\ln\zeta \tag{A.9c}$$

provided that $M=N=0$.

10. *Lorentz Transformation with Translation*
Equation: $(L_2+L_4)\Gamma=0$
Solution:

$$\Gamma=\Gamma(\chi), \qquad \chi=\zeta-\frac{E}{2}s^2 \tag{A.10}$$

provided that

$$b_{\alpha\nu}A^{\nu}=Ee_{\alpha} \tag{A.10'}$$

which implies that $N=0$.

11. *Dilation and Translation*
Equation: $(L_3+L_4)\Gamma=-2c$
Solution:

$$\Gamma=-2\ln(c\zeta+N) \tag{A.11}$$

12. *Proper Conformal and Lorentz Transformations with Translation*
Equation: $(L_1+L_2+L_4)\Gamma=-4\phi$

(a) if $e_{\lambda}=Ea_{\lambda}$ and $b_{\mu\nu}a^{\nu}=Ba_{\mu}$, which implies $|b_{\mu\nu}-B\varepsilon_{\mu\nu}|=0$ and $a^2=0$.
special solution,

$$\Gamma=f=-2\ln(s^2+E) \tag{A.12a}$$

general solution,

$$\Gamma=f+\bar{\Gamma}(\chi) \tag{A.12ai}$$

where $\chi=A(\phi-B)/(s^2+E)+K$.

(b) another solution:

$$\Gamma=-2\ln\left(\zeta-\tfrac{1}{2}Bs^2\right) \tag{A.12b}$$

provided that $b_{\mu\nu}A^{\nu}=BA_{\mu}$, which implies $|b_{\mu\nu}-B\varepsilon_{\mu\nu}|=0$ and $M=N=0$.

13. *Proper Conformal Transformation with Dilation and Translation*
Equation: $(L_1+L_3+L_4)^{\Gamma}=-2c-4\phi$
Solution:

(a) if $e_{\lambda}=Ea_{\lambda}$,

$$\Gamma=-2\ln\left[\phi+\frac{a^2}{c}(s^2+E)\right] \tag{A.13a}$$

(b) if

$$\Gamma=-2h\left(\bar{M}\zeta-M\bar{\zeta}\right) \tag{A.13b}$$

where $\zeta=A_{\alpha}x^{\alpha}$, $\bar{\zeta}=\bar{A}_{\alpha}x^{\alpha}$, $\bar{M}=a^{\lambda}\bar{\zeta}_{\lambda}$, $\bar{N}=e^{\lambda}\bar{\zeta}_{\lambda}$.

condition:

$$\bar{M}N = M\bar{N} \tag{A.13b$'$}$$

special case: $\zeta = \phi$, $\quad \bar{\zeta} = \theta$, $\quad M = a^2$, $\quad \bar{M} = N = \varepsilon$, $\quad \bar{N} = e^2$
then

$$\varepsilon^2 = a^2 e^2 \tag{A.13b$'$}$$

14. *Lorentz Transformation with Dilation and Translation*
Equation: $(L_2 + L_3 + L_4)\Gamma = -2c$
Solution:

$$\Gamma = -\frac{2c}{c-B}\ln[\zeta(c-B)+N] \tag{A.14}$$

provided that $b_{\mu\nu}A^{\nu} = BA_{\mu}$, which implies $|b_{\mu\nu} - \varepsilon_{\mu\nu}B| = 0$.

15. *Complete Conformal Group*
Equation: $(L_1 + L_2 + L_3 + L_4)\Gamma = -2c - 4\phi$
Let $b_{\mu\nu}A^{\nu} = BA_{\mu}$ and $b_{\mu\nu}\bar{A}^{\nu} = \bar{B}\bar{A}_{\mu}$, which imply $|b_{\mu\nu} - B\varepsilon_{\mu\nu}| = 0$ and $|b_{\mu\nu} - \bar{B}\varepsilon_{\mu\nu}| = 0$, where $\zeta = A_{\mu}x^{\mu}$, $\bar{\zeta} = \bar{A}_{\mu}x^{\mu}$.
Solution:

$$\Gamma = -2\ln\left(F\zeta + G\bar{\zeta} + h\right) \tag{A.15}$$

where $h = (G\bar{M} + FM)s^2 + (G\bar{N} + FN)$, F, G are constants, and

$$B = 2\frac{G}{F}\bar{N} + 2N, \qquad \bar{B} = -2\bar{M} - 2\frac{F}{G}M \tag{A.15$'$}$$

which requires that

$$4(M\bar{N} + \bar{M}N) = B\bar{B} + 2(\bar{M}B - N\bar{B}) \tag{A.15$''$}$$

Special case: if $\zeta = \phi$, $\bar{\zeta} = \theta$, then $h = (G\varepsilon + Fa^2)s^2 + (Ge^2 + F\varepsilon)$ and $B = \frac{2}{F}(Ge^2 + F\varepsilon)$, $\bar{B} = -\frac{2}{G}(G\varepsilon + Fa^2)$, which requires that

$$\frac{B\bar{B}}{4} + \frac{\varepsilon}{2}(B - \bar{B}) = \varepsilon^2 - a^2 e^2 \tag{A.15$'''$}$$

Note: Capital letters, such as A, B, etc., are constants and others are defined in Table I.

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